JOURNAL OF APPROXIMATION THEORY 37, 5–11 (1983)

Chebyshev Approximation by Rationals with Constrained Denominators

CHARLES B. DUNHAM*

Computer Science Department, University of Western Ontario, London, Ontario N6A 5B9, Canada

Communicated by E. W. Cheney

Received November 2, 1978

Approximating families of rational functions can be made nicer (tamed) by constraining the denominators below and above. Topological properties are improved, but characterization and uniqueness are more difficult for non-interior points.

Let X be a compact Hausdorff space and $C(X)$ the space of real continuous functions on X. For Y a closed subset of X define

$$\|g\|_Y = \sup\{|g(x)| : x \in Y\}.$$

Let $\{\phi_1, ..., \phi_n\}$ and $\{\psi_1, ..., \psi_m\}$ be linearly independent subsets of $C(X)$. Define for $A \in E_{n+m}$ (Euclidean $(n+m)$-space),

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

Let μ, ν be given elements of $C(X)$ such that $0 < \mu \leqslant \nu$ and define

$$C_{\mu,\nu} = \{A : \mu \leqslant Q(A, \cdot) \leqslant \nu\}, \qquad R_{\mu,\nu} = \{R(A, \cdot) : A \in C_{\mu,\nu}\}. \tag{1}$$

We will assume that $R_{\mu,\nu}$ is non-empty and will study approximation of $f \in C(X)$ by $R_{\mu,\nu}$ with respect to the above norm.

In most cases we will have μ and ν widely separated, but we do not exclude the possibility of equality at some points.

The primary reason for a study of $R_{\mu,\nu}$ is that approximation by admissible rationals R_G (rationals with denominators merely required to be >0, studied by Cheney in his text [3, Chap. 5]) is frequently unpleasant due to bad topological properties including non-closure of the parameter space

* First version written on sabbatical at the University of British Columbia.

0021-9045/83/010005–07$03.00/0

and convergence in parameters not implying uniform convergence [13, p. 76]. These lead to possible non-existence of best approximations, discontinuity of the Chebyshev operator, and failure of discretization [22]. Examination of examples of these bad features suggests that it is denominators going to zero that cause all of these problems. It might be thought that if denominators are bounded away from zero, that is, we require only

$$\varepsilon \leqslant Q(A, \cdot) \tag{2}$$

for fixed $\varepsilon > 0$, all these problems would disappear. This idea is a good one but not sufficient to solve the problems, as any rational with positive denominator can be made to satisfy (2) by multiplying all coefficients by a large constant. Thus if we are going to remove any of the difficulties, denominators must be bounded above as well as below, hence the bounds of (1).

It should be noted that only restriction (2) is given in [9]: however, perusal of other work of the authors of [9] shows that a normalization is also intended.

It should be noted that Kaufman and Taylor [21] consider a lower bound on denominators and an upper bound on denominator coefficients.

We first study the topological properties of $R_{\mu,\nu}$ to see if the difficulties above are removed and then study the characterization and uniqueness problems.

Topological Properties

It is seen that the set of coefficients $C_{\mu,\nu}$ for $R_{\mu,\nu}$ is closed and convex. For convenience we define the parameter norm

$$\|A\| = \max\{|a_i|: i = 1,..., n + m\}.$$

Definition. A closed subset Y of X is called *parameter bounding* if $\{\|A^k\|\} \to \infty$ implies $\|R(A^k, \cdot)\|_Y \to \infty$.

Lemma 1. *Let Y be a closed set on which $\{\phi_1,..., \phi_n\}$ is independent. Then Y is parameter bounding.*

Proof. The constraint (1) bounds the coefficients of $Q(A, x)$ in $C_{\mu,\nu}$ by a straightforward generalization of a result of Rice [13, p. 24]. Hence if $\{\|A^k\|\} \to \infty$, the coefficients of $P(A^k, \cdot)$ are unbounded. By the result cited in the previous sentence, we must have $\|P(A^k, \cdot)\|_Y \to \infty$. But for $x \in Y$,

$$|R(A^k, x)| = |P(A^k, x)|/Q(A^k, x)| \geqslant |P(A^k, x)|/\sup\{\nu(x): x \in Y\},$$

hence $\|R(A^k, \cdot)\|_Y \to \infty$.

LEMMA 2. *If* $\{A^k\} \to A \in C_{\mu,\nu}$, *then* $\{R(A^k, \cdot)\} \to R(A, \cdot)\}$ *uniformly on* X.

Proof. This classical result follows from $Q(A, \cdot) \geqslant \mu > 0$.

The above two lemmas imply that $R_{\mu,\nu}$ satisfies Young's condition [8; 13, pp. 26–27] and existence follows. The author's paper [4] establishes continuity of the Chebyshev operator where the best approximation is unique. Krabs [10] handles discretization with (1) holding only on the set Y of approximation. The author [8] handles discretization with (1) holding on all of X.

We have seen that the topological properties of $R_{\mu,\nu}$ are the best possible. We now consider the price we pay for them. First, as $R_{\mu,\nu}$ is a proper subset of admissible rationals R_G, best approximation by $R_{\mu,\nu}$ may not be as close. Furthermore, characterization and uniqueness results are not as simple.

CHARACTERIZATION AND UNIQUENESS

A key set in characterization is

$$M(Y, A) = \{x: |f(x) - R(A, x)| = \|f - R(A, \cdot)\|_Y, \, x \in Y\}.$$

The arguments of the author [5, p. 152] show that $R_{\mu,\nu}$ has the betweeness property. The arguments of Meinardus and Schwedt [11, p. 305; 12, p. 140] show that $R_{\mu,\nu}$ has asymptotic convexity, hence it also the second Kolmogorov property (K2) [2, p. 262]. Any one of these three properties implies regularity (= being a sun) [2, p. 262]. We have the Kolmogorov-type characterization:

THEOREM. *A necessary and sufficient condition for* $R(A, \cdot)$ *to be best on* Y *is that there exist no* $B \in C_{\mu,\nu}$ *with*

$$[f(x) - R(A, x)][R(B, x) - R(A, x)] > 0, \qquad x \in M(Y, A).$$

For some applications, equivalent but more convenient characterizations may be needed.

DEFINITION. $R(A, \cdot)$ is an *interior point* of $R_{\mu\nu}$ if $R(A, \cdot)$ can be expressed as $R(B, \cdot)$, $\mu < Q(B, \cdot) < \nu$.

Remark. The denominator $Q(A, \cdot)$ of a non-interior point must touch *both* μ and ν—if it only touched one, multiplying it by a constant slightly less than one or slightly greater than one would give a denominator strictly between μ and ν, hence $R(A, \cdot)$ would be an interior point.

It is easily shown by convexity or betweeness arguments that an interior point is best in $R_{\mu,\nu}$ if and only if it is best in R_G. Hence the more concrete characterizations of Cheney [3, pp. 159–160] or the author [8] for R_G can be used for interior points.

Conversely, it appears to be difficult to get a more specific characterization than the above theorem for non-interior points, even if we study very simple and fixed $R(A, \cdot)$, μ, ν. It is expected that characterizations based on the associated linear space (slightly different but equivalent in [3, 8]) do not apply, as there likely exists $\{A^k\} \to A$ non-interior with $R(A^k, \cdot) \notin R_{\mu,\nu}$.

THEOREM. *$\{A: \|f - R(A, \cdot)\|_Y \leqslant \eta, A \in C_{\mu,\nu}\}$ is a closed convex set.*

Proof. Use betweeness [5] and convexity. Strict quasi-convexity of the approximation problem follows from arguments of Barrodale [1].

The uniqueness problem for regular families has a formal solution in terms of zero-sign compatibility [2, p. 263; 5; 6]. Whether this can be easily applied is an open question. However, betweeness arguments show that if $R(A, \cdot)$ is an interior point, $R(A, \cdot)$ is uniquely best in $R_{\mu,\nu}$ if and only if it is best in admissible rationals R_G: thus all the uniqueness results for R_G [3, p. 164; 8] are applicable. In particular we have

THEOREM. *Let $R(A, \cdot)$ be an interior point and the associated linear space be a Haar subspace. Then $R(A, \cdot)$ is unique whenever it is best.*

The strong uniqueness theorem [3, p. 165] still holds for interior points as $R_{\mu,\nu} \subset R_G$.

Non-interior points may not be uniquely best even in approximation by ordinary rationals.

EXAMPLE. Let $X = [0, 1]$ and approximate by ratios of constants to nth degree polynomials, $n \geqslant 2$. Let $\mu = 1$ and $\nu = 2$. Let $f(0) = \frac{3}{2}$ and $f(1) = 0$. As $1 \leqslant Q(A, \cdot) \leqslant 2$, we must have $|R(A, 0)| \leqslant 2\,|R(A, 1)|$. It is easily seen from this that $1/(1+x)$ and $1/(1+x^2)$ are best to f on the set $\{0, 1\}$. f can be extended to $[0, 1]$ so that the error norm of both on X is the error norm on $\{0, 1\}$.

Non-uniqueness was expected by Krabs [10, p. 235], but no example was given.

Uniqueness may hold in the case $m = 2$.

THEOREM. *Let X be a closed finite interval $[\alpha, \beta]$. Let $\mu < \nu$. Approximate by ratios of polynomials of degree $n - 1$ to polynomials of degree 1. Best approximations by $R_{\mu,\nu}$ are unique.*

Proof. By previous results, as interior point which is best is unique. Hence non-uniqueness can occur only if there are two different non-interiorpoints $R(A, \cdot)$ and $R(B, \cdot)$ best. By betweeness or convexity, $R(C, \cdot) = R((A+B)/2, \cdot)$ is also best. Unless $Q(A, x) = Q(B, x) = \mu(x)$, $Q(C, x) > \mu(x)$. Unless $Q(A, x) = Q(B, x) = \nu(x)$, $Q(C, x) < \nu(x)$. Hence $Q(C, \cdot)$ touches both μ and ν only if $Q(A, \cdot)$ and $Q(B, \cdot)$ are equal at two points. If this is the case, we must have $Q(A, \cdot) \equiv Q(B, \cdot)$. But best approximation by ratios of polynomials of degree $n-1$ over $Q(A, \cdot)$ is unique. The only remaining possibility is that $Q(C, \cdot)$ touches only one or none of (μ, ν). But by the remark, $R(C, \cdot)$ is an interior point, hence approximation by $R(C, \cdot)$ is unique. We have a contradiction to $R(A, \cdot)$ and $R(B, \cdot)$ distinct and best.

Remark. Whether strong uniqueness holds when the best approximation is a non-interior point is open.

A useful property in approximation on a closed interval $[\alpha, \beta]$ is Rice's property of varisolvence (unisolvence of variable degree [20, pp. 3ff]). Let us consider the case in which we approximate by ordinary rational functions. Standard arguments show that unisolvence of the usual degree holds at $R(A, \cdot)$ if $R(A, \cdot)$ is an interior point. However varisolvence need not hold at non-interior points.

EXAMPLE. Let $X = [0, 1]$ and $\mu = 1$, $\nu = 2$. Approximate by ratios of constants to first degree polynomials. Let $R(A, x) = 1/(1+x)$. Constants in the range $[\frac{1}{2}, 1]$ touch $R(A, \cdot)$. Thus $R(A, \cdot)$ cannot have property Z [13, p. 71; 20, p. 3] of degree 1. It does have property Z of degree 2 by classical results.

As $B \in C_{\mu,\nu}$ implies $|R(B, 0)| \leqslant 2\,|R(B, 1)|$, there is no $B \in C_{\mu,\nu}$ with

$$R(B, 0) > R(A, 0) = 1, \qquad R(B, 1) < R(A, 1) = \tfrac{1}{2}$$

hence solvence of degree 2 does not hold at $R(A, \cdot)$.

If we let denominators be of higher degree in the example, we still get solvence of degree $\geqslant 2$ failing at $R(A, \cdot)$. The example can be generalized to any μ, ν for which non-constant approximations are in $R_{\mu,\nu}$.

It is an open question whether particular algorithms for best approximation by R_G can be readily adapted to maintain the constraint (1). The differential correction algorithm, both verions of which are discussed by Barrodale, Powell, and Roberts [14], is adaptable [18]. The convergence results [3, pp. 171–172; 14; 15] apply: it may be necessary for a rate of convergence to assume $R(A, \cdot)$ best is unique and an interior point, making the approximately family like R_G in a neighbourhood of $R(A, \cdot)$.

The linear inequality method [3, p. 170] is probably the most easily adaptable algorithm. We merely replace $-Q(x) \leqslant -1$ in Cheney's formula (1) by $-Q(x) \leqslant -\mu(x)$ and $Q(x) \leqslant \nu(x)$.

Loeb [19] gives two algorithms. The weighted minimax algorithm of Loeb [3, pp. 170–171] maintains *no* constraint on denominators and its behaviour for $R_{\mu,\nu}$ should be similar to its behaviour for R_G [16].

Generalizations

In real approximation, a natural generalization is to apply a transformation as in [8]. Using transformations preserving Young's condition [7, p. 61] we get the same topological theory. Using the transformations of [8] we get betweeness and a similar theory for characterization and uniqueness.

Limited extensions to complex approximation are possible. We can replace the constraint of (1) by $\mu \leqslant |Q(A, \cdot)| \leqslant \nu$ and we get a similar topological theory. But betweeness [17, pp. 731–732] may hold only if we assume a *real* denominator (a denominator whose argument is *fixed* at each point of X is equivalent) with $\mu \leqslant Q(A, \cdot) \leqslant \nu$.

With real denominators satisfying (1), betweeness or convexity arguments show that an interior point is best if and only if it is best in rationals with positive denominators [17, p. 728]. The uniqueness theory for interior points is the same as for rationals with positive denominators. The example of non-uniqueness for ordinary rationals with $m > 2$ and the uniqueness theorem for ordinary rationals with $m = 2$ apply to rationals with real denominator satisfying (1). It should be noted that uniqueness may not hold [17, p. 732] with $m = 3$ even if (1) is dropped. Transformations can be used, but those preserving betweeness appear to be restricted to those mapping straight line segments into straight line segments [17, p. 728].

An alternative way to restrict denominators is to replace (1) by

$$C'_{\mu,\nu} = \{A : \mu < Q(A, \cdot) < \nu\} \; R'_{\mu,\nu} = \{R(A, \cdot) : A \in C'_{\mu,\nu}\}; \qquad (1')$$

$R'_{\mu,\nu}$ has betweeness and asymptotic convexity as before. $R'_{\mu,\nu}$ has a nice characterization and uniqueness theory—exactly the same as for R_G. Unpleasant behaviour of limits, such as in continuity of the best approximation operator or discretization, is eliminated. The price we pay is that limits may not exist—we have just thrown away all coefficient vectors that could cause any kind of trouble. We can transform such rationals as in [8] to get a family with betweeness. The characterization and uniqueness theory is then the same as in [8].

Extension to complex approximation is possible. Betweeness holds for rationals whose denominators are required to be positive (hence real [23]) and satisfy (1′).

Acknowledgment

The author wishes to thank the referee for numerous improvements.

References

1. I. Barrodale, Best rational approximation and strict quasi-convexity *SIAM J. Numer. Anal.* **10** (1973), 8–12.
2. D. Braess, Geometrical characterizations for nonlinear uniform approximation, *J. Approximation Theory* **11** (1974), 260–274.
3. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
4. C. B. Dunham, Existence and continuity of the Chebyshev operator, *SIAM Rev.* **10** (1968), 444–446.
5. C. B. Dunham, Chebyshev approximation by families with the betweeness property, *Trans. Amer. Math. Soc.* **136** (1969), 151–157.
6. C. B. Dunham, Characterizability and uniqueness in real Chebyshev approximation, *J. Approx. Theory* **2** (1969), 374–383.
7. C. B. Dunham, The limit of best approximation on subsets, *Aequationes Math.* **9** (1973), 60–63.
8. C. B. Dunham, Transformed rational Chebyshev approximations, *J. Approx. Theory* **19** (1977), 200–204.
9. E. H. Kaufman, Jr., D. J. Leeming, and G. D. Taylor, A combined Remes-differential correction algorithm for rational approximation, *Math. Comp.* **32** (1978), 233–242.
10. W. Krabs, On discretization in generalized rational approximation *Abh. Math. Sem. Univ. Hamburg* **39** (1973), 231–244.
11. G. Meinardus and D. Schwedt, Nicht-lineare approximationen, *Arch. Rational Mech. Anal.* **17** (1964), 297–326.
12. G. Meinardus, "Approximation of Functions," Springer-Verlag, New York, 1967.
13. J. R. Rice, "Approximation of Functions," Vol. 1, Addison–Wesley, Reading, Mass., 1964.
14. I. Barrodale, M. J. D. Powell, and F. D. K. Roberts, The differential correction algorithm for rational l_∞ approx., *SIAM J. Numer. Anal.* **9** (1972), 493–504.
15. S. N. Dua and H. L. Loeb, Further remarks on the differential correction algorithm, *SIAM J. Numer. Anal.* **10** (1973), 123–126.
16. C. B. Dunham, The weighted minimax algorithm, *Utilitas Math.* **12** (1977), 247–253.
17. C. B. Dunham, Chebyshev approximation by families with the betweeness property, II, *Indiana Univ. Math. J.* **24** (1975), 727–732.
18. E. W. Cheney and H. L. Loeb, Two new algorithms for rational approximation, *Numer. Math.* **3** (1961), 72–75.
19. H. L. Loeb, Algorithms for Chebyshev approximations using the ratio of linear forms, *J. Soc. Indust. Appl. Math.* **8** (1960), 458–465.
20. J. R. Rice, "Approximation of Functions," Vol. 2, Addison–Wesley, Reading, Mass., 1969.
21. E. H. Kaufman, Jr. and G. D. Taylor, Uniform approximation by rational functions having restricted denominators, *J. Approx. Theory* **32** (1981), 9–26.
22. C. B. Dunham, Nonexistence of best rational approximations on subsets *J. Approx. Theory* **14** (1975), 160–161.
23. C. B. Dunham, Complex approximation versus real approximation, *Mathematica* **21** (1979), 127–130.